MATHEMATICA MORAVICA
VOL. **3** (1999), 97–104

CONTINUITY OF GENERAL J-CONVEX FUNCTIONS

Milan R. Tasković

Abstract. In this paper we continue the study of the general J--convex functions, which are introduced in our former paper (Tasković, Math. Japonica, **37** (1992), 367-372). We prove that if $D \subset \mathbb{R}^n$ a convex and open set, and if $f : D \to \mathbb{R}$ is a general J-inner function with the property of local oscillation in D, then it is continuous in D.

Since every J-convex function (also an additive function) is general J-inner function, we obtain as a particular case of the preceding statement the result of F. Bernstein and G. Doetsch.

1. Introduction and history

Let $D \subset \mathbb{R}^n$ be a convex and open set. A function $f : D \to \mathbb{R}$ is called **J-convex** if it satisfies Jensen's functional inequality

$$\text{(J)} \qquad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in D$. If the inequality in (J) for $x \neq y$ is sharp, f is called **strictly J-convex**. These functions were introduced (for $n = 1$) by J. W. Jensen [12], although functions satisfying similar conditions were already treated by Hadamard [8], Hermite [9], Hölder [10] and Stolz [23].

Basic properties of J-convex functions in the one-dimensional case were proved by Jensen himself and by Bernstein-Doetsch [3]. Generalizations to higher dimensions were made by Blumberg [4] and Mohr [17].

On can easily see that if a J-convex function is bounded on the open interval I that it is continuous in I, and if it is unbounded on I that it is unbounded on any subinterval on I. This means, if a J-convex function is bounded on any subinterval of I it is continuous in I, from Bernstein-Doetsch paper [3].

AMS (MOS) Subject Classification 1991. Primary: 49A40, 90C30. Secondary: 49A35, 90C48.

Key words and phrases: Convex functions, Quasiconvex functions, General convex functions, Extremal problems, Measurability, Continuity, General inner functions.

Fréchet [7] has proved that a measurable function which satisfies the following Cauchy's functional equation of addivity

$$f(x+y) = f(x) + f(y)$$

is continuous. Sierpiński [21] has proved that a measurable J-convex function is also continuous.

Various other proofs of the preceding statements were then supplied by Banach [2], Kac [13], Alexiewicz-Orlicz [1], Kuczma-Smital [14], Paganoni [19], Fischer-Slodkowski [6], Figiel [5], Kurepa [15], Seneta [20] and Steinhaus [22].

Ostrowski [18] has proved the statement according to which a J-convex function bounded on a set of the positive measure is continuous.

In the present paper we prove some analogous statements of the preceding type for general J-convex (inner) functions. With this statements we precision, correction and expand our the former results for general J-inner functions (Theorems 2 and 3 in [25]).

In our former paper, Tasković [24], has introduced the notion of general J-convex functions. A function $f : D \to \mathbb{R}$, where $\mathbb{R}$ denotes the real line and D is a convex subset of $\mathbb{R}^n$, is said to be **general J-convex** if there is a function $g : f(D)^2 \to \mathbb{R}$ such that

$$\text{(M)} \qquad f\left(\frac{x+y}{2}\right) \le \max\Big\{f(x),\ f(y),\ g\big(f(x),\ f(y)\big)\Big\}$$

for all $x, y \in D$. We notice that the set of all J-convex functions can be a proper subset of the set of all general J-convex functions.

On the other hand, recall that a function $f : D \to \mathbb{R}$ is said to be **general J-concave** if there is a function $g : f(D)^2 \to \mathbb{R}$ such that

$$\text{(N)} \qquad \min\big\{f(x),\ f(y),\ g\big(f(x),\ f(y)\big)\big\} \le f\left(\frac{x+y}{2}\right)$$

for all $x, y \in D$. If $f : D \to \mathbb{R}$ general J-convex and J-concave function, then f a **general J-inner function.**

We notice that the set of all J-convex and J-concave functions can be a proper subset of the set of all general J-inner functions.

2. Local boundedness of general J-inner functions

Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is called **locally bounded (locally bounded above, locally bounded below)** at a point $x_\circ \in D$ if there exists a neighbourhood $U \subset D$ of $x_\circ$ such that the function f is bounded (bounded above, bounded below) on U. We start with the following statements from Tasković [25].

Lemma 1. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-convex function for some bounded above function $g : f(D)^2 \to \mathbb{R}$. If f is locally bounded above at a point $x_\circ \in D$, then it is locally bounded above at every point $x \in D$.*

A brief proof of this statement may be found in Tasković [25]. We notice that when considered dually and analogously to the preceding Lemma 1, we obtain directly the following statement.

Lemma 2. (Dually of Lemma 1). *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-concave function for some bounded below function $g : f(D)^2 \to \mathbb{R}$. If f is locally bounded below at a point $x_\circ \in D$, then it is locally bounded below at every point $x \in D$.*

Thus, when we combine two preceding lemmas, we directly obtain the following statement for general J-inner functions.

Theorem 1. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-inner function for some bounded function $g : f(D)^2 \to \mathbb{R}$. If f is locally bounded at a point $x_\circ \in D$, then it is locally bounded at every point $x \in D$.*

A brief proof of this statement may be found in Tasković [25]. Since J-convex functions have support lines, thus, we notice that J-convex functions are, de facto, general J-inner functions. Thus we obtain directly the following statement.

Corollary 1. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a J-convex function. If f is locally bounded above at a point $x_\circ \in D$, then it is locally bounded at every point $x \in D$.*

3. Continuity of general J-inner functions

A statement of Bernstein-Doetsch [3] says that if $D \subset \mathbb{R}^n$ is a convex and open set, $f : D \to \mathbb{R}$ is a J-convex function, $T \subset D$ is open and nonempty, and f is bounded above on T, then f is continuous in D. Are there other sets T for general J-convex and J-inner functions with this property? In this section we will deal with such questions.

We begin with the following essential lemmas.

Lema 3. *Let $D \subset \mathbb{R}^n$ be a convex and open set. If $f : D \to \mathbb{R}$ is a general J-convex function, then there is a function $g : f(D)^n \to \mathbb{R}$ such that*

$$(1)\quad f\left(\frac{x_1 + \cdots + x_n}{n}\right) \le \max\Big\{f(x_1), \ldots, f(x_n),\ g\big(f(x_1), \ldots, f(x_n)\big)\Big\}$$

for every $n \in \mathbb{N}$ and for every $x_1, \ldots, x_n \in D$.

Induction shows that this statement holds. As an immediate consequence of Lemma 3 we obtain the following statement.

Lemma 4. *Let $D \subset \mathbb{R}^n$ be a convex and open set. If $f : D \to \mathbb{R}$ is a general J-convex function, then there is a function $g : f(D)^2 \to \mathbb{R}$ such that*

$$f(\lambda x + (1-\lambda)y) \le \max\Big\{f(x),\ f(y),\ g\big(f(x),\ f(y)\big)\Big\} \tag{2}$$

for every $x, y \in D$ and for every $\lambda \in \boldsymbol{Q} \cap [0,1]$, where $\boldsymbol{Q}$ denotes the set of rational numbers.

Proof. Let $\lambda = k/n$ for $n \in \mathbb{N}$ and $0 < k < n$. Put $x_1 = \cdots = x_k = x$ and $x_{k+1} = \cdots = x_n = y$. By (1) we obtain

$$f\left(\frac{kx + (n-k)y}{n}\right) \le \max\Big\{f(x),\ f(y),\ g(f(x), \dots, f(x),\ f(y), \dots, f(y))\Big\}$$

which is the same as (2). If $\lambda = 0$ or 1, the inequality (2) is trivial. The proof is complete.

We notice, if $f : D \to \mathbb{R}$ is general J-convex and continuous, then (2) holds for all real $\lambda \in [0,1]$.

In connection with the preceding, let $D \subset \mathbb{R}^n$. A general J-inner function $f : D \to \mathbb{R}$ for some function $g : f(D)^2 \to \mathbb{R}$ is called **locally oscillation** at a point $x_\circ \in D$ if there exists a neighbourhood $U \subset D$ of $x_\circ$ such that

$$\begin{aligned}&\max\Big\{f(x),\ f(y),\ g\big(f(x),\ f(y)\big)\Big\} - \\ &- \min\Big\{f(z),\ f(t),\ g\big(f(z),\ f(t)\big)\Big\} < C\|x - x_\circ\|\end{aligned} \tag{Os}$$

for some constant $C > 0$, and for all $x, y, z, t \in U$.

A mapping $f : D \to \mathbb{R}$ is with the **property of local oscillation** in D if it have the locally oscillation at every point of D.

We are now in a position to formulate the following statement with who we precision, correction and expand our Theorem 2 of [25].

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-inner function. If f with the property of local oscillation in D, then it is continuous in D.*

Proof. Let f be a locally oscillation at a point of D. Take an arbitrary $x_\circ \in D$. Thus there exists constant $r > 0$ such that $K(x_\circ, r) \subset D$ and (Os) for all $x, y, z, t \in K(x_\circ, r)$. For an arbitrary $x \in K(x_\circ, r)$ we have $\|x - x_\circ\| < r$, and consequently we can find a $\lambda \in \boldsymbol{Q} \cap (0,1)$ such that $\|x - x_\circ\| r^{-1} < \lambda < 2\|x - x_\circ\| r^{-1}$. Put

$$y = x_\circ + \lambda^{-1}(x - x_\circ), \quad z = x_\circ - \lambda^{-1}(x - x_\circ). \tag{3}$$

Hence we have $||y-x_\circ|| < r$, and $||z-x_\circ|| < r$, which means that $y, z \in \in K(x_\circ, r)$. Also, we obtain that $x = \lambda y + (1-\lambda)x_\circ$ and $x_\circ = (1+\lambda)^{-1}x + (1+ +\lambda)^{-1}\lambda z$. By Lemma 4, since f is general J-inner we have from (M) and (N)

$$\min\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\} \le f(x) \le \le \max\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\} \tag{4}$$

and

$$\min\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\} \le f(x_\circ) \le \le \max\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\}. \tag{5}$$

Since $x, y, z, x_\circ \in K(x_\circ, r)$ we obtain the following inequalities of the form

$$f(x) - f(x_\circ) \le \max\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\} - - \min\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\}$$

and

$$f(x) - f(x_\circ) \ge \min\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\} - - \max\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\}$$

Hence by (Os), since $x, y, z, x_\circ \in K(x_\circ, r)$, we obtain that

$$-C||x-x_\circ|| < f(x) - f(x_\circ) < C||x-x_\circ||. \tag{6}$$

The preceding inequalities (6) proves the continuity of f at a point $x_\circ \in D$. Whence, since $x_\circ \in D$ has been arbitrary, the continuity of f in D follows. The proof is complete.

In connection with the preceding proof, we have the following locally form of (Os). A general J-inner function $f : D \to \mathbb{R}$ for some function $g : : f(D)^2 \to \mathbb{R}$ is called **round locally oscillation** at a point $x_\circ \in D$ if there exists a neighbourhood $U \subset D$ of $x_\circ$ such that

$$A := \max\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\} - - \min\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\} < C||x-x_\circ|| \tag{Oa}$$

and

$$B := \min\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\} - - \max\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\} > -C||x-x_\circ|| \tag{Ob}$$

for some constant $C > 0$, and for all $x, y, z \in U$. A mapping $f : D \to \mathbb{R}$ is with the **property of round local oscillation** in D if it have the round locally oscillation at every point of D.

An immediate consequence of the preceding proof of Theorem 2 is the following statement.

Theorem 2a. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-inner function. If f with the property of round local oscillation in D, then it is continuous in D.*

This proof is totally analogous to the proof of the preceding statement.

From the preceding statement, we are now in a position to formulate the following consequence.

Corollary 2. (Bernstein-Doetsch [3]). *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a J-convex function. If f is locally bounded above at a point of D, then it is continuous in D.*

Proof. We notice that J-convex functions are general J-inner functions. Also, let f be a locally bounded at an arbitrary point $x_\circ \in D$. Thus there exist positive constants M and r such that $K(x_\circ, r) \in D$ and $|f(t)| \leq M$ for $t \in K(x_\circ, r)$. Then, the analogous to the preceding proof, for an arbitrary $x \in K(x_\circ, r)$ we have $||x - x_\circ|| < r$, and we can find a $\lambda \in Q \cap (0, 1)$ such that

$$||x - x_\circ|| r^{-1} < \lambda < 2||x - x_\circ|| r^{-1},$$

and that (3). Since f is a J-convex function we obtain from Lemma 4

$$f(x) \leq \lambda f(y) + (1 - \lambda) f(x_\circ), \quad f(x_\circ) \leq (1 + \lambda)^{-1} f(x) + \lambda(1 + \lambda)^{-1} f(z)$$

for all points $x, y, z \in K(x_\circ, r)$. Thus f satisfies the conditions (Oa) and (Ob), because

$$A \leq \lambda f(y) + (1 - \lambda) f(x_\circ) - f(x_\circ) = \lambda\big[f(y) - f(x_\circ)\big] < 4Mr^{-1}||x - x_\circ||$$

and

$$B \geq f(x) - (1 + \lambda)^{-1} f(x) - \lambda(1 + \lambda)^{-1} f(z) =$$
$$= \lambda(1 + \lambda)^{-1}\big[f(x) - f(z)\big] > -4Mr^{-1}||x - x_\circ||.$$

This means that f satisfy all the required hupotheses in Theorem 2a. Applying Theorem 2a to the mapping f we obtain that it is continuous in D. The proof is complete.

Further, we shall introduce the concept of round locally boundness in $D \subset \mathbb{R}^n$. A general J-inner function $f : D \to \mathbb{R}$ for some bounded above function $g : f(D)^2 \to \mathbb{R}$ with $M > 0$ is called **round locally bounded above** at a point $x_\circ \in D$ if there exists a neighbourhood $U \subset D$ of $x_\circ$ such that

$$\text{(Bo)} \qquad f(x) \leq M < \min\Big\{f(y),\ f(z),\ g\big(f(y),\ f(z)\big)\Big\} + C||x - x_\circ||$$

for some constant $C > 0$, and for all $x, y, z \in U$. A mapping $f : D \to \mathbb{R}$ is with the **property of round local bounded above** in D if it have the round locally bounded above at every point of D.

We are now in a position to formulate the following statement with who we precision, correction and expand our Theorems 2 and 3 of [25].

Theorem 3. *Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-inner function. If f with the property of round local bounded above in D, then it is continuous in D.*

Proof. Let f be a round locally bounded above at a point of D. Take an arbitrary $x_\circ \in D$. Thus there exist positive constants M, C and r such that (Bo) for all $x, y, z \in K(x_\circ, r) \subset D$. For an arbitrary $x \in K(x_\circ, r)$ we have $||x - x_\circ|| < r$, and consequently we can find a $\lambda \in Q \cap (0, 1)$ such that $||x - x_\circ||^{-1} < \lambda < 2||x - x_\circ||r^{-1}$. From (3) and by Lemma 4, since f is general J-inner we obtain (4) and (5) for all $x, y, z \in K(x_\circ, r)$. Hence by (Bo) we obtain

$$(7) \quad f(x) - f(x_\circ) \le M - \min\Big\{f(x),\ f(z),\ g\big(f(x),\ f(z)\big)\Big\} < C||x - x_\circ||$$

and

$$(8) \quad f(x) - f(x_\circ) \ge -\Big(M - \min\Big\{f(y),\ f(x_\circ),\ g\big(f(y),\ f(x_\circ)\big)\Big\}\Big) > -C||x - x_\circ||$$

for all $x, y, z \in K(x_\circ, r)$. The preceding inequalities (7) and (8) proves the continuity of f at a point $x_\circ \in D$. Whence, since $x_\circ \in D$ has been arbitrary, the continuity of f in D follows. The proof is complete.

Open problem. Let $D \subset \mathbb{R}^n$ be a convex and open set, and let $f : D \to \mathbb{R}$ be a general J-inner function. If f is locally oscillation at a point of D, does f have the property of local oscillation in D?

4. References

[1] A. Alexiewicz and W. Orlicz: *Remarque sur l'équation fonctionnelle $f(x + y) = f(x) + f(y)$*, Fund. Math., **33** (1945), 314-315.

[2] S. Banach: *Sur l'équation fonctionnelle $f(x + y) = f(x) + f(y)$*, Fund. Math., **1** (1920), 123-124.

[3] F. Bernstein and G. Doetsch: *Zur Theorie der konvexen Funktionen*, Math. Ann., **76** (1915), 514-526.

[4] H. Blumberg: *On convex functions*, Trans. Amer. Math. Soc., **20** (1919), 40-44.

[5] T. Figiel: *O równaniu funkcyjnym $f(x + y) = f(x) + f(y)$*, Wiadom. Mat., **11** (1969), 15-18.

[6] P. Fischer and Z. Slodkowski: *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc., **79** (1980), 449-453.

[7] M. Fréchet: *Pri la funkcia equacio $f(x+y) = f(x)+f(y)$*, Enseignement Math., **15** (1913), 390-393.

[8] J. Hadamard: *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171-215.

[9] Ch. Hermite: *Sur deux limites d'une intégrale définie*, Mathesis, **3** (1883), 82-83.

[10] O. Hölder: *Über einen Mittelwertssatz*, Nachr. Ges. Wiss. Göttingen 1889, 38-47.

[11] M. Hukuhara: *Sur la fonction convexe*, Proc. Japan Acad., **30** (1954), 683-685.

[12] J. L. W. V. Jensen: *Sur les fonctions convexes et les inégalitiés entre les valeurs moyennes*, Acta Math., **30** (1906), 179-193.

[13] M. Kac: *Une remarkue sur les équations fonctionnelles*, Comment. Math. Helv., **9** (1936/37), 170-171.

[14] M. Kuczma and J. Smital: *On measures connected with the Cauchy equation*, Aequationes Math., **14** (1976), 421-428.

[15] S. Kurepa: *Convex functions*, Glasnik Mat. Fiz. Astronom., **11** (1956), 89-93.

[16] J. H. B. Kemperman: *A general functional equation*, Trans. Amer. Math. Soc., **86** (1957), 28-56.

[17] E. Mohr: *Beitrag zur Theorie der konvexen Funktionen*, Math. Nachr., **8** (1952), 133-148.

[18] A. Ostrowski: *Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen*, Jahresber. Deutsch. Math. Ver., **38** (1929), 54-62.

[19] L. Paganoni: *Una estensione di un teorema di Steinhaus*, Ist. Lombardo Accad. Sci. Lett. Rend. A, **108** (1974), 262-273.

[20] E. Seneta: *Regular varying functions*, Springer Lecture Notes, **508** (1976).

[21] W. Sierpiński: *Sur l'équation fonctionnelle $f(x+y) = f(x)+f(y)$*, Fund. Math., **1** (1920), 116-122.

[22] H. Steinhaus: *Sur les distances des points des ensembles de measure positive*, Fund. Math., **1** (1920), 99-104.

[23] O. Stolz: *Grundzüge der Differential und Integralrechnung*, Vol. **1**, Teubner, Leipzig, 1893.

[24] M. R. Tasković: *General convex functions*, Math. Japonica, **37** (1992), 367-372.

[25] M. R. Tasković: *Measurability of general J-convex functions*, Math. Japonica, **41** (1995), 609-613.

Matematički fakultet
11000 Beograd, P.O. Box 550
Yugoslavia

Received January 7, 1997.